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On Local Linear Functionals for L-Splines*

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Quasi-interpolant fumnctionals for L-splines are constructed. With them as a tool, an explicit construction of LB-splines is given, as well as a quick proof of the existence and uniqueness of the expansion of an L-spline in an LB-spline series. Moreover, a necessary and sufficient condition for a function to generate a local linear functional that vanishes at all LB-splines but one, is obtained.

1. INTRODUCTION

We begin with some notations and definitions.

Let $k \in \mathbb{N}$, $\mathbf{t} := (t_i)$ nondecreasing (finite, infinite or biinfinite) with $t_i < t_{i+k}$, all i, and let

$$a := \inf\{t_i\}, \qquad b := \sup\{t_i\},$$

$$c_i := \max\{m : t_{i-m} = t_i\},$$

$$l_i := \max\{m : t_{i+m} = t_i\},$$

$$d_i := c_i + l_i + 1,$$

$$\mathrm{jump}_{t_i} f := f(t_i+) - f(t_i-).$$

Let $H_p^k(a, b)$ denote the space of functions which are k-fold integrals of functions in $L_p(a, b)$, $1 \leqslant p \leqslant \infty$. Further, let

$$L = \sum_{j=0}^{k} p_j D^{k-j}$$

be a nonsingular kth order differential operator, where $p_0 \equiv 1$, $p_j \in C^j(a, b)$ $(j = 1,..., k)$ and $D = d/dx$. Then the formal adjoint operator of L is

$$L^* = \sum_{j=0}^{k} (-1)^j D^j(p_{k-j} \cdot).$$

By N_L and N_{L^*} we denote the null spaces of L and L^*, respectively.

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Throughout this paper we suppose that a basis of N_L forms a locally Extended Tchebycheff System, namely, that the following condition:

(ET) The sum of multiplicities of g's zeros does not exceed $k-1$ on $[t_i, t_{i+k}]$ for any nonzero $g \in N_L$ and any i

is satisfied.

DEFINITION 1.1. A function S defined on (a, b) is called an L-spline with knots $\mathbf{t}$ if

(i) $S|_{(t_i, t_{i+1})} \in N_L|_{(t_i, t_{i+1})}$ for all i;

(ii) $\text{jump}_{t_i} S^{(\gamma)} = 0$ for all i and $\gamma < k - d_i$.

DEFINITION 1.2. $[i, j]$ is called the carrier of a nonzero L-spline S and $j-i$ is said to be its length if

(i) $S = 0$ outside $[t_i, t_j]$;

(ii) $\text{jump}_{t_i} S^{(\gamma)} = 0$ for $\gamma < k - l_i - 1$, but $\text{jump}_{t_i} S^{(k-l_i-1)} \neq 0$;

(iii) $\text{jump}_{t_j} S^{(\gamma)} = 0$ for $\gamma < k - c_j - 1$, but $\text{jump}_{t_j} S^{(k-c_j-1)} \neq 0$.

It should be noticed that for a given nonzero L-spline S its carrier $[i, j]$ is uniquely determined. For, if $[i', j']$ also satisfies (i), (ii) and (iii), then first, $t_i = t_{i'}$ (otherwise, say, $t_i > t_{i'}$, then (i) implies that $\text{jump}_{t_{i'}} S^{(\gamma)} = 0$ for all γ, which conflicts with (ii)); second, (ii) yields $i = i'$; similarly, $j = j'$.

DEFINITION 1.3. A nonzero L-spline S with minimum carrier is called an LB-spline.

Here "S has a minimum carrier" means that there are no other nonzero L-splines whose carrier is a proper subset of the carrier of S.

The purpose of this paper is to extend some results of polynomial B-splines to LB-splines. In Section 2 we construct quasi-interpolant functionals for LB-splines. In Section 3 we give an explicit construction of LB-splines. In Section 4 we obtain the expansion of an L-spline in an LB-spline series with the quasi-interpolant functionals as a tool. In Section 5 we extend de Boor's results about local linear functionals to LB-splines.

2. QUASI-INTERPOLANT

For a fixed integer i, let μ_m be the functional given by

$$\begin{aligned} \mu_m(f) &= f^{(m-i-1)}(t_m) \qquad \text{when} \quad m = i+1, \ldots, i+l_i; \\ &= f^{(c_m)}(t_m) \qquad \text{when} \quad m \geqslant i + l_i + 1. \end{aligned} \tag{2.1}$$

LEMMA 2.1. *There exists a nonzero function $u_i(x) \in N_{I}$, which satisfies*

$$\mu_m(u_i) = 0, \qquad m = i+1, \ldots, i+k-1.$$

Moreover, such a function is unique up to a constant factor.

Proof. Let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be a basis of N_{I}. It is easily seen that the function

$$u_i(x) = \begin{vmatrix} \mu_{i+1}(\varphi_1) \, \mu_{i+2}(\varphi_1) & \cdots & \mu_{i+k-1}(\varphi_1) & \varphi_1(x) \\ \mu_{i+1}(\varphi_2) \, \mu_{i+2}(\varphi_2) & \cdots & \mu_{i+k-1}(\varphi_2) & \varphi_2(x) \\ \vdots \quad \vdots & & \vdots & \vdots \\ \mu_{i+1}(\varphi_k) \, \mu_{i+2}(\varphi_k) & \cdots & \mu_{i+k-1}(\varphi_k) & \varphi_k(x) \end{vmatrix} \tag{2.2}$$

satisfies

$$\mu_m(u_i) = 0, \qquad m = i+1, \ldots, i+k-1.$$

We claim that

$$u_i(x) \neq 0 \qquad \text{when } x \in (t_j, t_{j+1}), \ j = i, \ldots, i+k-1.$$

Suppose to the contrary that there exists some $x \in (t_j, t_{j+1})$ $(j = i, \ldots, i+k-1)$ for which $u_i(x) = 0$. Then we can find $\gamma_1, \gamma_2, \ldots, \gamma_k$, of which at least one is not zero, so that

$$\gamma_1 \mu_j(\varphi_1) + \gamma_2 \mu_j(\varphi_2) + \cdots + \gamma_k \mu_j(\varphi_k) = 0, \qquad j = i+1, \ldots, i+k-1$$

and

$$\gamma_1 \varphi_1(x) + \gamma_2 \varphi_2(x) + \cdots + \gamma_k \varphi_k(x) = 0.$$

Let $\varphi = \gamma_1 \varphi_1 + \gamma_2 \varphi_2 + \cdots + \gamma_k \varphi_k$. Then φ is not a zero function, and the sum of the multiplicities of φ's zeros exceeds $k-1$. This contradicts the condition (ET).

Suppose now that another function v has the same property as u_i. We have to show that there exists a constant c such that $v = cu_i$. There are the following two possibilities:

(i) $t_i < t_{i+1}$. In this case it follows from the condition (ET) that $u_i(t_i) \neq 0$ and $v(t_i) \neq 0$. If we put $c = v(t_i)/u_i(t_i)$, then the function $v - cu_i \in N_{I}$, and the sum of multiplicities of its zeros would exceed or equal k, hence $v - cu_i = 0$, that is, $v = cu_i$.

(ii) $t_i = t_{i+1}$. Thus we know that $u_i^{(l_i)}(t_i) \neq 0$ and $v^{(l_i)}(t_i) \neq 0$ in view of the condition (ET). A similar demonstration gives that $v = cu_i$ for $c = v^{(l_i)}(t_i)/u_i^{(l_i)}(t_i)$.

The determinant on the right-hand side of (2.2) is abbreviated to

$$\det\begin{pmatrix}\mu_{i+1}, \mu_{i+2}, \ldots, \mu_{i+k-1}, x\\ \varphi_1, \varphi_2, \ldots, \varphi_{k-1}, \varphi_k\end{pmatrix}.$$

COROLLARY 2.1. *If $\psi_1, \psi_2, \ldots, \psi_k$ is another basis of N_{L^*}, then there exists a constant c such that*

$$\det\begin{pmatrix}\mu_{i+1}, \mu_{i+2}, \ldots, \mu_{i+k-1}, x\\ \psi_1, \psi_2, \ldots, \psi_{k-1}, \psi_k\end{pmatrix} \equiv c \cdot \det\begin{pmatrix}\mu_{i+1}, \mu_{i+2}, \ldots, \mu_{i+k-1}, x\\ \varphi_1, \varphi_2, \ldots, \varphi_{k-1}, \varphi_k\end{pmatrix}. \quad (2.3)$$

Now we consider Lagrange's Formula [9]. If $f \in H_p^k(\alpha, \beta)$ and $g \in H_q^k(\alpha, \beta)$, where $1/p + 1/q = 1$, then

$$\int_\alpha^\beta (Lf)g\,dx = \int_\alpha^\beta (L^*g)f\,dx + W(f, g; x)\Big|_\alpha^\beta, \qquad -\infty < \alpha \leqslant \beta < \infty, \quad (2.4)$$

where

$$\begin{aligned} W(f, g; x) := \sum_{\gamma=0}^{k} \{f^{(\gamma-1)}(x)[p_{k-\gamma}(x)g(x)] &- f^{(\gamma-2)}(x)[p_{k-\gamma}(x)g(x)]' + \cdots \\ &+ (-1)^{\gamma-1}f(x)[p_{k-\gamma}(x)g(x)]^{(\gamma-1)}\}. \end{aligned} \quad (2.5)$$

In particular, if $f|_{(\alpha,\beta)} \in N_L$ and $g|_{(\alpha,\beta)} \in N_{L^*}$, then it follows from (2.4) that

$$W(f, g; \alpha+) = W(f, g; \beta-). \quad (2.6)$$

Taking an L-spline S as f and taking u_i as g in (2.5), we have, for $x \notin \mathbf{t}$,

$$\begin{aligned} W(S, u_i; x) = \sum_{\gamma=0}^{k} \{S^{(\gamma-1)}(x)[p_{k-\gamma}(x)\, u_i(x)] &- S^{(\gamma-2)}(x)[p_{k-\gamma}(x)\, u_i(x)]' \\ &+ \cdots + (-1)^{\gamma-1}S(x)[p_{k-\gamma}(x)\, u_i(x)]^{(\gamma-1)}\}. \end{aligned} \quad (2.7)$$

If $t_i < t_m < t_{i+k}$, then

$$u_i(t_m) = u_i^{(d_m-1)}(t_m) = 0,$$

$$\text{jump}_{t_m} S = \cdots = \text{jump}_{t_m} S^{(k-d_m-1)} = 0,$$

hence

$$W(S, u_i; t_m+) = W(S, u_i; t_m-), \quad (2.8)$$

and $W(S, u_i, t_m)$ is defined to have this common value. On the other hand, we have, for any $\xi, \eta \in (t_i, t_{i+k})\backslash\mathbf{t}$,

$$W(S, u_i; \eta) - W(S, u_i; \xi) = \sum_{\xi \le t_m \le \eta} |W(S, u_i; t_m +) - W(S, u_i; t_m -)|.$$

Therefore,

$$W(S, u_i; \eta) - W(S, u_i; \xi) = 0,$$

that is,

$$W(S, u_i; \eta) = W(S, u_i; \xi), \qquad \text{for any } \xi, \eta \in (t_i, t_{i+k})\backslash\mathbf{t}. \tag{2.9}$$

In light of (2.8), (2.9) remains valid for ξ and/or $\eta \in (t_i, t_{i+k}) \cap \mathbf{t}$. We conclude that $W(S, u_i; \cdot)$ is identically equal to a constant in (t_i, t_{i+k}).

DEFINITION 2.1. By $\mathscr{S}(L; \mathbf{t})$ we denote the space of all L-splines with knots $\mathbf{t}$. The linear functional

$$\lambda_i : S \to W(S, u_i; \xi), \qquad t_i < \xi < t_{i+k}, \tag{2.10}$$

which acts on the space $\mathscr{S}(L; \mathbf{t})$ is called a quasi-interpolant functional.

THEOREM 2.1. *If S is an L-spline with $[m, n]$ as its carrier, then*

(1°) $\lambda_i S = 0$ *when* $m > i$;

(2°) $\lambda_i S \neq 0$ *when* $m = i$;

(3°) $\lambda_i S = 0$ *when* $n < i + k$;

(4°) $\lambda_i S \neq 0$ *when* $n = i + k$.

Proof. (1°) If $t_m > t_i$, we take $\xi \in (t_i, t_m)$, then

$$\lambda_i S = W(S, u_i; \xi) = 0$$

since $S \equiv 0$ on (t_i, t_m). In the case of $t_m = t_i$, from

$$S(t_i) = S'(t_i) = \cdots = S^{(k-l_i-1)}(t_i) = 0,$$

$$u_i(t_i) = u_i'(t_i) = \cdots = u_i^{(l_i-1)}(t_i) = 0$$

it follows that

$$\lambda_i S = W(S, u_i; t_i +) = 0.$$

(2°) Suppose to the contrary that the statement $\lambda_i S = 0$ holds. There are two cases:

(i) $t_i < t_{i+1}$. Substituting $W(S, u_i; t_i+) = 0$ and

$$S(t_i) = S'(t_i) = \cdots = S^{(k-2)}(t_i) = 0$$

into (2.7), we obtain

$$S^{(k-1)}(t_i+)\, u_i(t_i) = 0,$$

but $u_i(t_i) \neq 0$ in terms of the condition (ET) and $S^{(k-1)}(t_i+) \neq 0$, so we get a contradiction.

(ii) $t_i = t_{i+1}$. In this case,

$$S(t_i) = S'(t_i) = \cdots = S^{(k-l_i-2)}(t_i) = 0,$$
$$u_i(t_i) = u_i'(t_i) = \cdots = u_i^{(l_i-1)}(t_i) = 0.$$

Combining it with (2.7), we have

$$S^{(k-l_i-1)}(t_i+)\, u_i^{(l_i)}(t_i) = 0,$$

which contradicts the fact that $S^{(k-l_i-1)}(t_i) \neq 0$ and $u_i^{(l_i)}(t_i) \neq 0$.

We can similarly prove (3°) and (4°).

COROLLARY 2.2. *The length of any nonzero L-spline S is at least k.*

In fact, if $[m, n]$ *is the carrier of S and* $n - m < k$, *then* (2°) *of Theorem* 2.1 *implies* $\lambda_m S \neq 0$, *but* (3°) *implies* $\lambda_m S = 0$.

3. THE CONSTRUCTION OF LB-SPLINES

There are other papers which deal with the construction of LB-splines (cf. Jerome and Schumaker [6]), but the construction given here is particularly suited for the development of the quasi-interpolant functionals. Further, we emphasize that LB-splines are entirely determined by the operator L and are independent of the choice of N_L's basis.

LEMMA 3.1. *If* $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ *is a basis in* N_{L^*}, *then there exists a basis* $\{\chi_1, \chi_2, \ldots, \chi_k\}$ *in* N_L *such that, for* $l = 0, 1, \ldots, j$,

$$\sum_{i=1}^{k} \varphi_i^{(l)}(\xi)\, \chi_i^{(j-l)}(\xi) \equiv \begin{cases} 0 & \text{when } j = 0, 1, \ldots, k-2; \\ (-1)^l & \text{when } j = k-1. \end{cases} \tag{3.1}$$

The functions (χ_i) are the adjunct functions for the (φ_i); see [8; 669]. Let

$$\begin{aligned} G(x, \xi) &= \sum_{i=1}^{k} \varphi_i(\xi)\, \chi_i(x), & x \geqslant \xi, \\ &= 0, & x < \xi. \end{aligned} \tag{3.2}$$

Clearly, $G(x, \xi)$ is Green's function for the operator L with side conditions:

$$y(\alpha) = y'(\alpha) = \cdots = y^{(k-1)}(\alpha) = 0, \alpha \leqslant x, \xi.$$

Now we define functionals v_m as follows:

$$v_m(f) := \begin{cases} f^{(m-i)}(t_m), & m = i, \dots, i + l_i; \\ f^{(c_m)}(t_m), & m \geqslant i + l_i + 1. \end{cases} \tag{3.3}$$

It is easily seen that

$$K_m(x) \equiv v_m(G(x, \cdot)), \qquad m = i, i+1, \dots$$

are L-splines. By (3.1) we have

(i) For $m = i, \dots, i + l_i$,

$$\begin{aligned} \text{jump}_{t_m} K_m^{(\gamma)} &= 0 & \gamma &< k - 1 - m + i; \\ &= (-1)^{m-i}, & \gamma &= k - 1 - m + i. \end{aligned}$$

(ii) For $m \geqslant i + l_i + 1$,

$$\begin{aligned} \text{jump}_{t_m} K_m^{(\gamma)} &= 0, & \gamma &< k - 1 - c_m; \\ &= (-1)^{c_m}, & \gamma &= k - 1 - c_m. \end{aligned}$$

Thus the function

$$M_i(\varphi_1, \dots, \varphi_k; x) := \begin{vmatrix} v_i(\varphi_1) & v_i(\varphi_2) & \cdots & v_i(\varphi_k) & v_i(G(x, \cdot)) \\ v_{i+1}(\varphi_1) & v_{i+1}(\varphi_2) & \cdots & v_{i+1}(\varphi_k) & v_{i+1}(G(x, \cdot)) \\ \vdots & \vdots & & & \\ v_{i+k}(\varphi_1) & v_{i+k}(\varphi_2) & \cdots & v_{i+k}(\varphi_k) & v_{i+k}(G(x, \cdot)) \end{vmatrix} \tag{3.4}$$

is an L-spline with $[i, i+k]$ as its carrier. The M_i's length equals k, but by Corollary 2.2 the length of any nonzero L-spline is not less than k, so we have already proved the main part of the following theorem.

THEOREM 3.1. *$M_i(\varphi_1, \varphi_2, \dots, \varphi_k; x)$ given by (3.4) is an LB-spline. Moreover each LB-spline M can be represented as*

$$M = \text{const} \cdot M_i(\varphi_1, \dots, \varphi_k; \cdot) \qquad \textit{for some } i.$$

Proof. Suppose M's carrier is $[i, j]$. By Corollary 2.2 we know $j \geqslant i + k$; on the other hand, we have $j - i \leqslant k$ by the definition of LB-splines, so $j = i + k$. By Definition 1.2,

$$\text{jump}_{t_i} M_i^{(k-l_i-1)} \neq 0 \qquad \text{and} \qquad \text{jump}_{t_i} M^{(k-l_i-1)} \neq 0.$$

Let

$$c := \text{jump}_{t_i} M^{(k-l_i-1)} / \text{jump}_{t_i} M^{(k-l_i-1)}.$$

Then $M - cM_i$ would have a carrier which is a proper subset of $[i, j]$. Applying Corollary 2.2 again to this case, we have $M - cM_i = 0$, that is, $M = cM_i$.

COROLLARY 3.1. *For any two bases of* $N_{L^*} = \{\varphi_1, \varphi_2, ..., \varphi_k\}$ *and* $\{\psi_1, \psi_2, ..., \psi_k\}$, *there exists a nonzero constant c such that*

$$M_i(\psi_1, \psi_2, ..., \psi_k; x) \equiv c \cdot M_i(\varphi_1, \varphi_2, ..., \varphi_k; x).$$

4. LB-SPLINE SERIES

It follows directly from Theorem 2.1 that

THEOREM 4.1. *For i, j integers, let M_j be an LB-spline with $[t_j, t_{j+k}]$ as its carrier, and let λ_i be a quasi-interpolant functional given by* (2.10). *Then*

$$\lambda_i M_j \neq 0$$

if and only if $i = j$.

COROLLARY 4.1. *For any open set I*, $\{M_i; \operatorname{supp} M_i \cap I \neq \varnothing\}$ *is linearly independent on I.*

Proof. Suppose

$$\sum_{\operatorname{supp} M_i \cap I \neq \varnothing} \gamma_i M_{i|_I} = 0.$$

Letting the functional $\lambda_i = W(\cdot, u_i; \xi_i)$, where $\xi_i \in \operatorname{supp} M_i \cap I$, act on the foregoing equation, we obtain

$$\gamma_i = 0 \qquad \text{for all } i \text{ such that } \operatorname{supp} M_i \cap I \neq \varnothing.$$

COROLLARY 4.2. $\overline{\operatorname{supp}(\sum_i \gamma_i M_i)} = \overline{\bigcup_{\gamma_i \neq 0} \operatorname{supp} M_i}$.

Proof. The relation

$$\overline{\operatorname{supp} \sum_i \gamma_i M_i} \subset \overline{\bigcup_{\gamma_i \neq 0} \operatorname{supp} M_i}$$

is obvious. Conversely, suppose $\tau \in \overline{\operatorname{supp} M_j}$ for some $j, \gamma_j \neq 0$, but $\tau \notin \overline{\operatorname{supp} \sum_i \gamma_i M_i}$. Then we can choose some τ_j inside $\operatorname{supp} M_j$ so that $\tau_j \notin \overline{\operatorname{supp} \sum_i \gamma_i M_i}$. If we put $\lambda_j = W(\cdot, u_j : \tau_j)$, then

$$\lambda_j \sum_i \gamma_i M_i = 0,$$

hence $\gamma_j = 0$, which is a contradiction.

With the help of quasi-interpolant functionals we can obtain the following existence and uniqueness theorem about LB-spline series expansion. The proof is omitted here because it is similar to the proof in [3].

THEOREM 4.2. *Any L-spline S can be represented as a series of LB-splines:*

$$S = \sum_i a_i M_i;$$

moreover, this representation is unique.

5. LOCAL LINEAR FUNCTIONALS

It is remarkable that a series of important results about polynomial splines and approximation theory could be obtained through de Boor's local linear functionals (see [2]). In this section we give a necessary and sufficient condition that a function generate a local linear functional that vanishes at all LB-splines but one. This extends de Boor's corresponding results (see [2, Lemma 3.1]) to LB-splines.

DEFINITION 5.1. *If*

$$f^{(c_m)}(t_m) = g^{(c_m)}(t_m), \qquad \forall\, m, \tag{5.1}$$

then we say that f "agrees with" g at $\mathbf{t}$ *and write*

$$f|_{\mathbf{t}} = g|_{\mathbf{t}}.$$

Suppose, for i integers, M_i are LB-splines, and u_i are given by (2.2). Without loss we may assume $\lambda_i M_j = \delta_{ij}$, where λ_i is given by (2.10). Let $n := i + k - c_{i+k}$. Then

$$t_i \leqslant t_{n-1} < t_n = \cdots = t_{i+k}.$$

Let

$$\begin{aligned} u_i^+ &= 0, \qquad \text{if} \quad t < (t_{n-1} + t_n)/2; \\ &= u_i, \qquad \text{if} \quad t \geqslant (t_{n-1} + t_n)/2. \end{aligned} \tag{5.2}$$

We have

THEOREM 5.1. *$h_i \in L_q(a, b)$ satisfies*

$$\int h_i M_j = \delta_{ij}, \qquad \textit{all } i, j,$$

if and only if $h_i = -L^*f$ for some $f \in H_q^k(a, b)$ with $f|_{\mathbf{t}} = u_i^+|_{\mathbf{t}}$.

Proof. "If" part. Suppose $f|_{\mathbf{t}} = u_i^+|_{\mathbf{t}}$. We have, for any L-spline S,

$$W(S, f; t_m+) = W(S, f; t_m-), \qquad m \leqslant n-1, \tag{5.3}$$

and

$$W(S, f-u_i; t_m+) = W(S, f-u_i; t_m-), \qquad m \geqslant n. \tag{5.4}$$

In view of Lagrange's formula we have

$$\begin{aligned} \int_{t_m}^{t_{m+1}} (L^*f)\, S\, dx &= \int_{t_m}^{t_{m+1}} (LS) f\, dx - W(S, f; x)\Big|_{t_m+}^{t_{m+1}-} \\ &= W(S, f; t_m+) - W(S, f; t_{m+1}-), \qquad t_m < t_{m+1}, \end{aligned}$$

hence

$$\int (L^*f)\, M_j\, dx = \sum_{t_j \leqslant t_m < t_{m+1} \leqslant t_{j+k}} [W(M_j, f; t_m+) - W(M_j, f; t_{m+1}-)]. \tag{5.5}$$

Let us separate consideration of the following three possibilities.

(i) $t_{j+k} \leqslant t_{n-1}$. In this case, it follows from (5.3) and (5.5) that

$$\int (L^*f)\, M_j\, dx = W(M_j, f; t_j+) - W(M_j, f; t_{j+k}-),$$

but

$$W(M_j, f; t_j+) = 0, \qquad W(M_j, f; t_{j+k}-) = 0 \tag{5.6}$$

by (2.5) and the definition of LB-splines, so that $\int (L^*f)\, M_j\, dx = 0$.

(ii) $t_j \geqslant t_n$. We have, similarly,

$$W(M_j, f-u_i; t_j+) = 0, \qquad W(M_j, f-u_i; t_{j+k}-) = 0. \tag{5.7}$$

We rewrite (5.5) as

$$\begin{aligned}\int (L^*f) M_j dx &= \sum_{t_j \leqslant t_m < t_{m+1} \leqslant t_{j+k}}' [W(M_j, f-u_i; t_m+) - W(M_j, f-u_i; t_{m+1}-)] \\ &+ \sum_{t_j \leqslant t_m < t_{m+1} \leqslant t_{j+k}}' [W(M_j, u_i; t_m+) - W(M_j, u_i; t_{m+1}-)].\end{aligned}$$

The first sum is equal to zero by (5.4) and (5.7). To calculate the second sum we resort to Lagrange's Formula and obtain

$$\begin{aligned}&\sum_{t_j \leqslant t_m < t_{m+1} \leqslant t_{j+k}}' [W(M_j, u_i; t_m+) - W(M_j, u_i; t_{m+1}-)] \\ &= \sum_{t_j \leqslant t_m < t_{m+1} \leqslant t_{j+k}}' \left[\int_{t_m}^{t_{m+1}} (L^*u_i) M_j dx - \int_{t_m}^{t_{m+1}} (LM_j) y_i dx\right] = 0. \end{aligned} \tag{5.8}$$

(iii) $t_{j+k} > t_{n-1}$ and $t_j < t_n$. Thus $t_j \leqslant t_{n-1} < t_n \leqslant t_{j+k}$ must occur. Let

$$\sum_{t_j \leqslant t_m < t_{m+1} \leqslant t_{j+k}}' [W(M_j, f; t_m+) - W(M_j, f; t_{m+1}-)] = \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{5.9}$$

where

$$\begin{aligned}\Sigma_1 := &\sum_{t_j \leqslant t_m < t_{m+1} \leqslant t_{n-1}}' [W(M_j, f; t_m+) - W(M_j, f, t_{m+1}-)] \\ &+ W(M_j, f; t_{n-1}+),\end{aligned} \tag{5.10}$$

$$\begin{aligned}\Sigma_2 := &-W(M_j, f-u_i; t_n-) \\ &+ \sum_{t_n \leqslant t_m < t_{m+1} \leqslant t_{j+k}}' [W(M_j, f-u_i; t_m+) - W(M_j, f-u_i; t_{m+1}-)],\end{aligned} \tag{5.11}$$

$$\begin{aligned}\Sigma_3 := &-W(M_j, u_i; t_n-) \\ &+ \sum_{t_n \leqslant t_m < t_{m+1} \leqslant t_{j+k}}' [W(M_j, u_i; t_m+) - W(M_j, u_i; t_{m+1}-)].\end{aligned} \tag{5.12}$$

It follows from (5.3), (5.4), (5.6) and (5.7) that

$$\Sigma_1 = 0, \qquad \Sigma_2 = 0.$$

A demonstration similar to that in (5.8) gives

$$\sum_{t_n \leq t_m < t_{m+1} \leq t_{j+k}} [W(M_j, u_i; t_m+) - W(M_j, u_i; t_{m+1}-)] = 0.$$

Finally we have

$$\int (L^*f) M_j \, dx = \Sigma_1 + \Sigma_2 + \Sigma_3 = -W(M_j, u_i; t_{n+1}-) = -\delta_{ij},$$

that is,

$$\int h_i M_j = \delta_{ij}.$$

This completes the proof of "if" part.

The proof of "only if" part is based on the following lemma.

LEMMA 5.1. (1°) *If* $f^{(l_s)}(t_s) = 0$ $(s = j, j+1, ..., j+l_j)$ *and* $W(M_{j-1}, f; t_{j-1}+) = 0$, *then* $f^{(l_{j-1})}(t_{j-1}) = 0$.

(2°) *If* $f^{(c_s)}(t_s) = 0$ $(s = j, j-1, ..., j-c_j)$ *and* $W(M_{j+1}, f; t_{j+1}-) = 0$, *then* $f^{(c_{j+1})}(t_{j+1}) = 0$.

Proof. It suffices to prove (1°), because the proof of (2°) is similar. There are two possibilities.

(i) $t_{j-1} < t_j$. In this case,

$$M_{j-1}(t_{j-1}) = M'_{j-1}(t_{j-1}) = \cdots = M_{j-1}^{(k-2)}(t_{j-1}) = 0, \qquad M_{j-1}^{(k-1)}(t_{j-1}+) \neq 0,$$

so by (2.5) we have $M_{j-1}^{(k-1)}(t_{j-1}+) f(t_{j-1}) = W(M_{j-1}, f; t_{j-1}+) = 0$; hence $f(t_{j-1}) = 0$.

(ii) $t_{j-1} = t_j$. Putting

$$M_{j-1}(t_{j-1}) = M'_{j-1}(t_{j-1}) = \cdots = M_{j-1}^{(k-l_{j-1}-2)}(t_{j-1}) = 0, \quad M_{j-1}^{(k-l_{j-1}-1)}(t_{j-1}) \neq 0$$

and

$$f(t_{j-1}) = \cdots = f^{(l_{j-1}-1)}(t_{j-1}) = 0$$

in the place of the expression (2.5) for $W(M_{j-1}, f; t_{j-1}+)$, we obtain $f^{(l_{j-1})}(t_{j-1}) = 0$.

Now we proceed with the proof of the necessity. If $h_i \in L_q(a, b)$ is such a function that $\int h_i M_j = \delta_{ij}$, all j, then there exists an $f \in H_q^k(a, b)$ such that $-L^*f = h_i$ and

$$f^{(l_s)}(t_s) = 0, \qquad s = i, i+1, ..., n-1; \tag{5.13}$$

$$f^{(c_s)}(t_s) = u_i^{(c_s)}(t_s), \qquad s = n, ..., i+k-1. \tag{5.14}$$

To prove $f|_{\mathbf{t}} = u_i|_{\mathbf{t}}$, that is to prove

$$f^{(l_s)}(t_s) = 0 \qquad \text{for all} \quad s \leqslant n-1, \tag{5.15}$$

$$f^{(c_s)}(t_s) = u_i^{(c_s)}(t_s) \qquad \text{for all } s \geqslant n, \tag{5.16}$$

we proceed by induction on s. We only need to prove (5.16), because the proof of (5.15) is similar. Suppose (5.16) is true for s such that $n \leqslant s \leqslant j-1$, where $j \geqslant i+k$. Consider the integral $\int M_{j-k}(L^*f)\,dx$. Calculate its value by (5.9)–(5.12). It is easily seen that the contribution of Σ_1 is zero, the contribution of Σ_2 is $-W(M_{j-k}, f-u_i; t_j-)$, and the contribution of Σ_3 is $-\delta_{i,j-k}$. On the other hand, $\int M_{j-k}(L^*f)\,dx = -\int M_{j-k}h_i\,dx = -\delta_{i,j-k}$; therefore,

$$W(M_{j-k}, f-u_i; t_j-) = 0.$$

Resorting to Lemma 5.1, we obtain

$$f^{(c_j)}(t_j) = u_i^{(c_j)}(t_j).$$

This completes the proof of the "only if" part, and so of the theorem.

COROLLARY 5.1. *If* $[\alpha, \beta] \subseteq [t_i, t_{i+k}]$, *and if* $f \in H_q^k[\alpha, \beta]$ *satisfies the following conditions:*

(i) $f^{(\gamma)}(\alpha) = 0$, $\gamma = 0, 1, \ldots, k-1$;

(ii) $f^{(\gamma)}(\beta) = u_i^{(\gamma)}(\beta)$, $\gamma = 0, 1, \ldots, k-1$;

(iii) $f^{(\gamma)}(t_j) = 0$, $\gamma = 0, 1, \ldots, k-d_j-1$ *for* $t_j \in (\alpha, \beta)$;

then h_i *determined by* $h_i = -L^*f$ *has support* $[\alpha, \beta]$ *and*

$$\int h_i M_j = \delta_{ij} \textit{ for all } j.$$

6. REMARKS

1. As de Boor [1] pointed out, it is unnecessary to assume that $(\phi_j)^k$ is a Chebyshev system in order to construct L-splines of local support. Jerome and Schumaker [6] did such a construction. Here, we shall follow de Boor [1] and give a brief description. For a fixed i, let μ_m be defined as in (2.1), and let $j \geqslant i+1$ be the least integer such that $\operatorname{span}(\mu_{i+1}, \ldots, \mu_j) \cap N_{I_i}^{\perp} \neq \varnothing$. If (ET) holds, then $j-i=k$; otherwise $j-i<k$. Thus there is (up to a constant factor) one and only one L-spline with $[i,j]$ as its carrier. Since our argument relies only on local properties of the operator L, most of the results

of this paper remain true without assumption (ET), except that some LB-splines might have a carrier of length less than k.

2. The purpose of introducing "carrier" instead of "support" is to avoid confusion. Such a confusion happens easily. For example, in [10, Theorem 2.2], the statement that B_i is the unique L-spline satisfying $B_i(x) > 0$ for $y_i < x < y_{i+m}$ and $B_i(x) = 0$ for $x < y_i, y_{i+m} < x$ is not true. For, if $y_i = y_{i+1} < y_{i+m} = y_{i+m+1}$, then there are two linearly independent LB-splines with $[i, i+m]$ and $[i+1, i+m+1]$ as their carriers, respectively, but they have the same support $[y_i, y_{i+m}]$! In this respect, I believe that Definition 1.3 is the most appropriate.

3. In Section 3, I have no intention to give a "new" construction of LB-splines. Such constructions have been made by several authors. Among them, Karlin [7] and Jerome [5] are worth mentioning. In fact, as Theorem 3.1 states, any construction for LB-splines is substantially unique. Our emphasis is placed on the role that local linear functionals play in spline functions.

4. After I finished this paper, I was made aware of K. Scherer and L. L. Schumaker [10]. They treat the same subject as that here. Though my approach to local linear functionals is more direct and explicit, they give some applications which I have not touched on in this paper. There is some hope that explicit numerical bounds for the norms of the linear functionals will be obtained, at least when L is an operator with constant coefficients. However, it would require more effort.

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